Hyperbolic Functions.

Ex– If $\cos(x+iy) = \cos\alpha + i\sin\alpha$, prove that

$$\cosh 2y + \cos 2x = 2$$

Solution:— We have, $\cos(x+iy) = \cos x \cos(iy) - \sin x \sin(iy)$

$$= \cos x \cosh y - i \sin x \sinh y$$

From the given condition, we have

$$\cos x \cosh y - i\sin x \sinh y = \cos\alpha + i\sin\alpha$$

Equating the real and imaginary parts, we get

$$\cos x \cosh y = \cos\alpha \quad \text{———— (1)}$$

and

$$-\sin x \sinh y = \sin\alpha \quad \text{———— (2)}$$

We need to eliminate α from (1) and (2).

Squaring (1) and (2), and then adding

$$\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y = 1$$

$$\Rightarrow 2\cos^2 x \cosh^2 y + 2\sin^2 x \sinh^2 y = 2$$

$$\Rightarrow (1+\cos 2x)\frac{1}{2}(1+\cosh 2y) + (1-\cos 2x)\frac{1}{2}(\cosh 2y - 1) = 2$$

$$\Rightarrow (1+\cos 2x)(1+\cosh 2y) - (1-\cos 2x)(1-\cosh 2y) = 4$$

$$\Rightarrow (1+\cos 2x + \cosh 2y + \cos 2x \cosh 2y) -$$

$$-(1 - \cos 2x - \cosh 2y + \cos 2x \cosh 2y) = 4.$$

$$\Rightarrow 2\cos 2x + 2\cosh 2y = 4$$

$$\therefore \cos 2x + \cosh 2y = 2. \quad \underline{\text{Proved.}}$$

Ex-2: Prove that $\sinh^{-1}(\tan\theta) = \log(\sec\theta + \tan\theta)$

Solution:— We have $\sinh^{-1} w = \log(w + \sqrt{w^2+1})$

$$\therefore \sinh^{-1}(\tan\theta) = \log(\tan\theta + \sqrt{\tan^2\theta + 1})$$

$$= \log(\tan\theta + \sqrt{\sec^2\theta})$$

$$= \log(\tan\theta + \sec\theta)$$

$$= \log(\sec\theta + \tan\theta)$$

—— Proved

Ex- Separate $\cos^{-1}(\cos\theta + i\sin\theta)$ into real and imaginary parts.

Solution:- Let $\cos^{-1}(\cos\theta + i\sin\theta) = A + iB$

$$\therefore \cos(A+iB) = \cos\theta + i\sin\theta$$

$$\Rightarrow \cos A \cos iB - \sin A \cdot \sin(iB) = \cos\theta + i\sin\theta$$

$$\Rightarrow \cos A \cosh B - i\sin A \sinh B = \cos\theta + i\sin\theta$$

Equating real and imaginary parts from both sides, we get

$$\cos A \cosh B = \cos\theta \quad \text{---} (1)$$

$$\sin A \sinh B = -\sin\theta \quad \text{---} (2)$$

We could have successively eliminated A and B from (1) and (2) and could have then separately expressed A and B in terms of θ, but instead we proceed as follows

Squaring (1) and (2) and then adding, we get

$$\cos^2 A \cosh^2 B + \sin^2 A \sinh^2 B = 1$$

$$\Rightarrow (1-\sin^2 A)\cosh^2 B + \sin^2 A(\cosh^2 B - 1) = 1$$

$$\Rightarrow \cosh^2 B - \sin^2 A \cosh^2 B + \sin^2 A \cosh^2 B - \sin^2 A = 1$$

$$\Rightarrow \cosh^2 B - \sin^2 A = 1$$

$$\Rightarrow \sin^2 A = \cosh^2 B - 1$$

$$\Rightarrow \sin^2 A = \sinh^2 B \qquad \therefore \sinh B = -\sin A$$

Now, from (2)

$$\sin A \sinh B = -\sin\theta$$

$$\Rightarrow -\sin A \cdot \sin A = -\sin\theta \Rightarrow -\sin^2 A = -\sin\theta$$

$$\Rightarrow \sin A = \sqrt{\sin\theta} \quad \therefore A = \sin^{-1}(\sqrt{\sin\theta})$$

Again, from (2)

$$\sin A \sinh B = -\sin\theta$$

$$\Rightarrow -\sinh B \cdot \sinh B = -\sin\theta$$

$$\Rightarrow \sinh^2 B = \sin\theta \Rightarrow \sinh B = \sqrt{\sin\theta}$$

$$\Rightarrow B = \sinh^{-1}(\sqrt{\sin\theta}) \quad \therefore B = \log\left(\sqrt{\sin\theta} + \sqrt{1+\sin\theta}\right)$$

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